Quantization of the undouble

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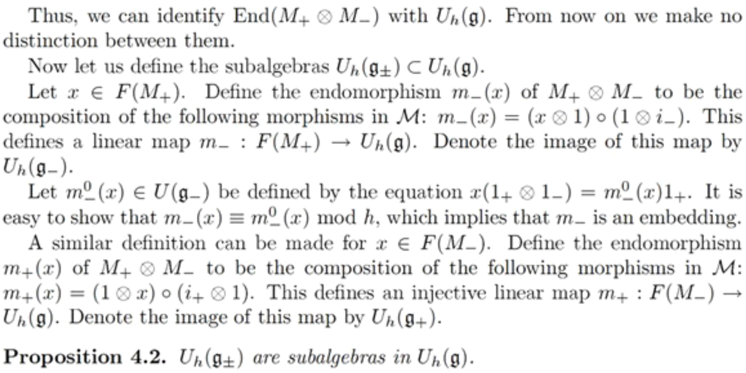
See also page 178 of Etingof-Schiffmann.

$+ :=$ head killer
$- :=$ tail killer.



Thus, we can identify $\mathrm{End}(M_+ \otimes M_-)$ with $U_h(\mathfrak{g})$. From now on we make no distinction between them.

Now let us define the subalgebras $U_h(\mathfrak{g}_\pm) \subset U_h(\mathfrak{g})$.

Let $x \in F(M_+)$. Define the endomorphism $m_-(x)$ of $M_+ \otimes M_-$ to be the composition of the following morphisms in $\mathcal{M}$: $m_-(x) = (x \otimes 1) \circ (1 \otimes i_-)$. This defines a linear map $m_- : F(M_+) \to U_h(\mathfrak{g})$. Denote the image of this map by $U_h(\mathfrak{g}_-)$.

Let $m_-^0(x) \in U(\mathfrak{g}_-)$ be defined by the equation $x(1_+ \otimes 1_-) = m_-^0(x)1_+$. It is easy to show that $m_-(x) \equiv m_-^0(x) \bmod h$, which implies that m_- is an embedding.

A similar definition can be made for $x \in F(M_-)$. Define the endomorphism $m_+(x)$ of $M_+ \otimes M_-$ to be the composition of the following morphisms in $\mathcal{M}$: $m_+(x) = (1 \otimes x) \circ (i_+ \otimes 1)$. This defines an injective linear map $m_+ : F(M_-) \to U_h(\mathfrak{g})$. Denote the image of this map by $U_h(\mathfrak{g}_+)$.

Proposition 4.2. *$U_h(\mathfrak{g}_\pm)$ are subalgebras in $U_h(\mathfrak{g})$.*

$x : M_+ \otimes M_- \longrightarrow M_+$

$$\begin{tikzcd}
M_+ \otimes M_- \arrow[d, "1\otimes i_-"] \arrow[ddd, bend right=60, "m_-(x)"'] \\
M_+ \otimes (M_- \otimes M_-) \arrow[d, "\Phi^{-1}"] \\
(M_+ \otimes M_-) \otimes M_- \arrow[d, "x\otimes 1"] \\
M_+ \otimes M_-
\end{tikzcd}$$

Question. Given $x \in U(\mathfrak{g}_-) \cong F(M_+)$, $y \in U(\mathfrak{g}) \cong M_+ \otimes M_-$,

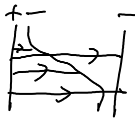
Find $m_-(x)(y) \in M_+ \otimes M_- \cong U(\mathfrak{g})$.

Answer. $x : 1_+ \otimes 1_- \longmapsto x \cdot 1_+$

$$y \longmapsto \Delta(y)\, 1_+ \otimes 1_- \xmapsto{1\otimes i_-} \Delta^3(y)\, 1_+ \otimes 1_- \otimes 1_-$$

$$\xmapsto{\Phi^{-1}} \Delta^3(y) \underbrace{\Phi^{-1} 1_{+--}}_{\approx 1_{+--} \text{ in co-commutative case}}$$

$$\xmapsto{x \otimes 1}$$



$\sim \Phi^{-1} 1_{+--}$

Recycling.

$$\xmapsto{x\otimes 1}$$

$$\Delta(y)\cdot(x\otimes 1)\cdot 1_+ \otimes 1_- \xmapsto{\phi^{-1}} m_-(x)(y) \in U(\mathfrak{g})$$

$$\parallel$$

$$\Delta(y)\cdot\Delta(x)\cdot 1_+ \otimes 1_-$$

$$\parallel$$

$$\Delta(yx)\cdot 1_+ \otimes 1_- \xrightarrow{\phi^{-1}} yx$$

So it looks like $m_-(x)(y) = yx$.